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The effect of non-linearity at the free surface on flow past a submerged cylinder

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Plane potential flow past a circular cylinder beneath a free surface under gravity is investigated in order to determine the importance or otherwise of non-linear effects from the free-surface boundary condition. It is shown that non-linear second-order corrections to the first-order linearized expressions for the wave-induced forces on the cylinder are considerably larger than second-order effects which are present even with a linear free-surface condition. Further evidence for the importance of non-linearity is presented in the form of streamline plots of the first-order solution showing strange behaviour at wave crests.

1. Introduction

It is common practice for workers on mathematical water-wave problems associated with ships to assume that their problems, usually irrotational potential flows past a fixed or moving rigid body at or near a free surface under gravity, are linear. That is, they accept the usual approximation (Lamb 1932, p. 363) to the Bernoulli equation which gives a linearized free-surface boundary condition on a known plane surface for the velocity potential. The common justification for this is that the non-linear free-surface condition is so intractable that there appears little hope of progress unless some simplifications are made, but nevertheless one should be quite clear under what circumstances the linearization process has a rational justification and should not attempt to apply it to physical situations where the phenomena are essentially non-linear.

There are three important situations of relevance to ship problems when linearization may be given formal justification, namely:

(A) A finite body making small (strictly also slow) oscillations about a state of rest. There can be little argument about this case, for clearly by making the disturbing motion sufficiently gentle we can make the waves produced smaller and flatter, which are the conditions required for the validity of the linearization process.

(B) A 'small' body in arbitrary motion at the free surface. For instance, if the ship looks like a vertical knife blade (a thin or 'Michell' ship), a formal successive approximation scheme can be used, an important property of which is that the boundary condition on the ship surface as well as that on the free surface must be linearized and applied on a simplified hull (in the case of a thin ship, on the centre plane). But here one meets with an objection from the naval architect who denies that he makes ships like knives, so how can it be sensible to apply boundary conditions on the centre plane? The temptation then is to go back to the *exact* boundary condition on the *actual* ship hull but still keep the linearized free-surface condition, since this would retain the benefits of linearity while avoiding the naval architect's objection. This procedure is inconsistent, however, and we should not expect it to give any better results than the first linearized approximation. There has, nevertheless, been some responsible consideration given to whether in numerical terms some worthwhile improvement in accuracy might be achieved, and one purpose of the present paper is to show that in a related problem *this is not so.*

(C) A finite body in arbitrary motion at a submergence large compared with its own dimensions. Clearly in this case also the body will produce small waves merely because it is a relatively distant disturbance. As distinct from case (B) it is now correct to satisfy the boundary condition on the body exactly for the first approximation, since the first approximation near the body is obviously the flow past the body in an infinite fluid without a free surface. For the second approximation consistency arguments still arise, however. Is it more important to take account of second approximations to the Bernoulli equation at the free surface, or on the other hand to include modifications to the flow due to the fact that the singularity distribution which generates the body in an infinite fluid no longer does so exactly in a fluid with a linearized free surface? Both of these effects are of second order in the ratio *body size/depth of submergence*, and we shall show circumstances in which the *first* is the more important effect.

2. Formulation of the problem

The problem to be discussed is of class (C), and has a long history (see Wehausen & Laitone, 1960, p. 574, for a list of references). Suppose we have a fixed circular cylinder of radius a with its centre at a depth h below the undisturbed level of the free surface. A stream of uniform velocity U at infinity flows past the cylinder normal to its axis, so that the flow is entirely two-dimensional and is described by a total complex potential

$$Uf = U(\phi + i\psi),$$

whose derivative is the complex conjugate of the flow-velocity vector. In the complex z-plane ($z = x+iy$) we take the origin at the centre of the circle and the y axis vertically upward so that $y = h$ is the undisturbed free-surface level. The fluid flows from left to right with $f \to z$ at infinity upstream, i.e. as $x \to -\infty$.

The disturbance to the free stream due to the presence of the cylinder causes waves on the free surface. If the cylinder is sufficiently deeply submerged these waves will be small, and we can build up the exact non-linear solution for the potential by solving a sequence of linear problems. As a consequence of wave formation there will be drag and lift forces on the cylinder even in the absence of viscosity and of circulation around the cylinder, and the calculation of these wave forces is our principal aim. Since the potential is determined as a series of terms which arise from linearized problems, the forces will also be found in the form of series, which converge, if at all, for sufficiently deep submergence. Convergent or not, the series may be viewed as asymptotic for large submergence, and from this viewpoint we calculate only the first two terms in order to examine the relative importance of two contributing effects which are both formally of second order.

If the acceleration due to gravity is g, then a characteristic length scale for the problem is U^2/g (in fact this is $1/2\pi$ times the wavelength of the waves produced by the disturbing cylinder), and it is convenient henceforth to consider all lengths to be measured in units of U^2/g. In other words, if z is the original co-ordinate we define a non-dimensional co-ordinate $z' = z(g/U^2)$ but immediately dispense with the dashes. In these units $U^2/g = 1$ and the waves are of length 2π. The quantities a and h are now also measured in terms of this length scale (e.g. now $a =$ (actual radius). (g/U^2)) and can in principle be of any order of magnitude except that a/h should be small. However, it will be convenient at a later stage to view a itself as a small parameter with $h = O(1)$, this being the physical range in which wave effects are most important.

On the cylinder
$$\psi = 0 \quad \text{on} \quad |z| = a, \tag{2.1}$$
while on the unknown instantaneous free surface
$$y = h + \eta(x), \tag{2.2}$$
the boundary conditions are
$$\psi = h \tag{2.3}$$
and
$$\eta = \tfrac{1}{2} - \tfrac{1}{2}|f'(z)|^2. \tag{2.4}$$
The linearized free-surface conditions have been given by many authors, e.g. Wehausen & Laitone (1960, pp. 465, 466, 471). If it is possible to write for the exact potential f an asymptotic series of the form
$$f = z + f_1 + f_2 + \ldots, \tag{2.5}$$
where the terms of the series are of diminishing order of magnitude in some sense, then the linearized conditions are
$$\mathscr{R}(f_1' + if_1) = 0, \tag{2.6}$$
$$\mathscr{R}(f_2' + if_2) = -\tfrac{1}{2}|f_1'|^2 + i(f_1'' + if_1')\mathscr{I}(f_1), \tag{2.7}$$
etc., where all quantities are evaluated on the equilibrium free surface $y = h$. The corresponding approximations to the wave height are
$$\eta_1 = -\psi_1, \tag{2.8}$$
$$\eta_2 = -\psi_2 + \eta_1^2, \tag{2.9}$$
etc. In general we may write for the nth approximation
$$\mathscr{R}(f_n' + if_n) = p_n(x), \tag{2.10}$$
where $p_n(x)$ is a complicated expression involving all f_m, $m < n$, and thus is a known function at the time when f_n is to be determined. In particular
$$p_1(x) = 0, \tag{2.11}$$
$$p_2(x) = -\tfrac{1}{2}|f_1'|^2 + i\psi_1(f_1'' + if_1'). \tag{2.12}$$
In physical terms $p_2(x)$ may be interpreted as the pressure distribution due to the first-order wave system, and the second-order flow f_2 is in part a flow produced by this pressure as a forcing agent.

3. The first approximation

The first approximation in the case of flow past a circle was given by Lamb in 1913 (Lamb 1932, p. 410). One simply replaces the circle by the dipole potential, modified so as to satisfy the linearized free-surface condition (2.6), obtaining

$$f_1 = \frac{a^2}{z} - \frac{a^2}{z-2ih} - 2a^2 i \int_0^\infty \frac{dk}{k-1} e^{-ik(z-2ih)}, \tag{3.1}$$

where the k-integration contour passes over the pole at $k = 1$. Clearly a^2/z represents an ordinary dipole at the centre of the circle and is the exact solution to the problem for a circle in an infinite fluid. The remaining terms, which tend to zero as a/h tends to zero and are regular everywhere in $\mathscr{I}z < 2h$, consist (Havelock 1926) of an image reversed dipole above the free surface plus a trailing 'tail' comprising a distribution of dipoles of constant strength but sinusoidally varying direction along the line $y = 2h$, $x > 0$.

By the substitutions

$$u = -i(k-1)(z-2ih), \quad \zeta = i(z-2ih),$$

the integral in (3.1) may be expressed in terms of the exponential integral

$$\mathrm{Ei}^{-}(\zeta) = \int_{-\infty-i0}^{\zeta} du \frac{e^u}{u}, \tag{3.2}$$

defined in the complex ζ plane cut along the *negative* real axis. The notation is that of Jahnke–Emde (1945); $\mathrm{Ei}^{-}(\zeta)$ has the expansion

$$\mathrm{Ei}^{-}(\zeta) = i\pi + \gamma + \log \zeta + \sum_{m=1}^{\infty} \frac{\zeta^m}{m m!} \tag{3.3}$$

in the above cut ζ plane, differing from other versions of the exponential integral (Jahnke–Emde 1945, p. 2) only in the presence or non-presence of the term $i\pi$, which in the present case leads to waves behind and not in front of the obstacle. Thus the first approximation to the total potential is

$$f = z + \frac{a^2}{z} - \frac{a^2}{z-2ih} + 2ia^2 e^{-2h+iz} \mathrm{Ei}^{-}(-2h-iz). \tag{3.4}$$

Numerical values for the potential or stream function are easily obtained by use of the series (3.3) for the exponential integral, which converges fast enough to enable quite a wide area of the plane to be mapped with present day computers. Streamlines and the first-order wave height η_1 from (2.8) are exhibited in figure 1 for the case $h = 2$, $a = 1$. This is a case in which we expect that the disturbance produced by the circle is too severe for the potential (3.4) to be valid as the first term in a convergent series representing the exact potential. However, this extreme case is used since the two principal phenomena of interest are exaggerated sufficiently to be visible and obvious; similar features occur at a reduced scale (or else move outside the region of physical interest) for more gentle disturbances.

The most peculiar feature of the indicated streamlines is the fact that behind the dipole some of them break up at wave crests into something resembling splashes. This demonstrates clearly the inadequacy of simple linearization for this severe disturbance; presumably the exact non-linear solution would involve highly non-sinusoidal or even breaking waves. Notice that a naive use of the linearized formula (2.8) for the wave height would not lead us to suspect that the

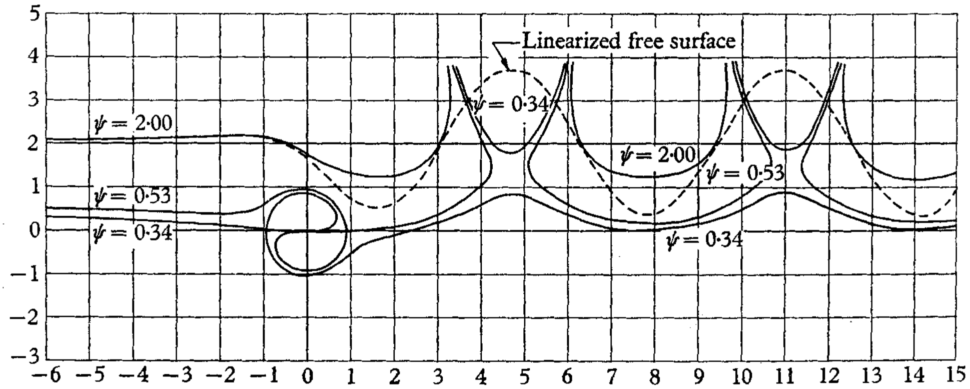

FIGURE 1. Streamlines associated with the linearized potential for two-dimensional flow past a submerged dipole.

streamlines were unreasonable. The dashed curve represents the linearized wave height, which is continuous and ultimately sinusoidal and is essentially distinct from the streamline ($\psi = 2{\cdot}00$) to which it is asymptotic far upstream (except that the two curves must touch whenever they cross the equilibrium free surface $y = 2$). The 'splash' phenomenon can be explained using the asymptotic form of the potential (3.4) far downstream (i.e. a free stream plus a linearized wave); by this means we can show that the streamline $\psi =$ constant cannot cross vertical lines through the wave crests unless $a^2 e^{-2h+\psi} < 1/4\pi e = 0{\cdot}0293$, which is a condition satisfied by only the streamline $\psi = 0{\cdot}34$ of those plotted in figure 1. For smaller a or larger h the critical streamline moves upwards until eventually all physically interesting streamlines are continuous.

The second important feature of figure 1 is that no closed body is generated by this first approximation, and in particular that the front and rear stagnation points are on different streamlines. This would be expected from a prior knowledge of the character of the second approximation, since (for instance) Havelock (1926) finds a contribution which can be interpreted as an oblique dipole at the centre. That is, the dipole of the first approximation has the wrong magnitude and direction and cannot possibly generate a closed body in combination with a uniform stream. In fact it appears likely that at no finite order of approximation is a closed body generated, which may serve as a warning to those seeking to use *inverse* methods to calculate ship-like bodies generated by given source distributions, but which need not deter us from pursuing the present *direct* problem further.

4. The Wehausen scheme

Further approximations have been calculated by a number of workers, although with the exception of the work of Bessho (1957) the first-order free-surface condition (2.6) is used for all orders of approximation, leading to an inconsistent linear problem. By an image method Havelock gave the second approximation in 1928 and, in a remarkably ingenious later paper (1936), was able to construct a complete formal solution to the wholly linear problem. In this paper he gave curves for the forces on the cylinder including contributions from approximations up to 6th order; however, we shall show that even his 2nd approximation is in general less important than the non-linear contribution from the pressure $p_2(x)$ of equation (2.12), which is ignored by Havelock.

Perhaps the most satisfactory way of tackling the wholly linear problem is that of Wehausen (Wehausen & Laitone 1960, p. 574). First let us define two operators which map certain classes of analytic functions onto each other. The 'Milne-Thomson' operator $\mathscr{M}$ is always applied to a function F analytic everywhere below the (undisturbed) free surface and yields a function analytic everywhere outside the cylinder by the formula

$$\mathscr{M}F(z) = \bar{F}(a^2/z), \tag{4.1}$$

as in the usual Milne-Thomson circle theorem (Milne-Thomson 1949, p. 149). Conversely the 'Kochin' operator $\mathscr{K}$ is always applied to a function G analytic everywhere outside the cylinder and yields a function analytic everywhere beneath the free surface by the formula

$$\mathscr{K}G(z) = \frac{1}{2\pi i}\oint d\zeta \bar{G}(\zeta - ih).\left[-\frac{1}{z-ih-\zeta} + 2i\,e^{-i(z-ih-\zeta)}\,\mathrm{Ei}^-(i(z-ih-\zeta))\right], \tag{4.2}$$

where the integral is around any closed path *above* the free surface which completely encloses the image of the cylinder in the free surface. As an example, if $G = 1/z$ we have by the residue theorem that

$$\mathscr{K}1/z = -\frac{1}{z-2ih} + 2i\,e^{-i(z-2ih)}\,\mathrm{Ei}^-(i(z-2ih)) \tag{4.3}$$

$$= \sum_{m=0}^{\infty} \frac{(-i)^{m+1}q_m}{m!} z^m$$

$$= -iq_0 - q_1 z + \tfrac{1}{2}iq_2 z^2 + \ldots, \tag{4.4}$$

where for later use we have expanded in a Taylor series about the origin, introducing the same coefficients $q_m = q_m(h)$ as were used by Havelock (1936), namely

$$q_m = \frac{m!}{(2h)^{m+1}} + 2\left\{\frac{(m-1)!}{(2h)^m} + \frac{(m-2)!}{(2h)^{m-1}} + \ldots + \frac{1}{2h} - e^{-2h}\mathrm{Ei}^-(2h)\right\}. \tag{4.5}$$

These $\mathscr{M}$ and $\mathscr{K}$ transformations have the properties that

$$F + \mathscr{M}F$$

satisfies the exact boundary condition (2.1) at the cylinder, i.e. is real on $|z| = a$, while

$$G + \mathscr{K}G$$

satisfies the homogeneous linearized free-surface condition (2.6). Now we construct the following sequence of operations:

$$\left.\begin{array}{lll} G_0 = 0 & F_0 = z & f_0 = G_0+F_0 \\ G_1 = \mathscr{M}F_0 & F_1 = \mathscr{K}G_1 & f_1 = G_1+F_1 \\ G_2 = \mathscr{M}F_1 & F_2 = \mathscr{K}G_2 & f_2 = G_2+F_2 \\ \vdots & \vdots & \vdots \\ G_n = \mathscr{M}F_{n-1} & F_n = \mathscr{K}G_n & f_n = G_n+F_n. \end{array}\right\} \tag{4.6}$$

Since $f_n = G_n+F_n = G_n+\mathscr{K}G_n$, each f_n satisfies the homogeneous free-surface condition (2.6). Wehausen proves that the f_n are of decreasing order of magnitude if a/h is sufficiently small and that the sequence

$$f = f_0+f_1+f_2+\dots$$

converges. Since also

$$f = \sum_0^\infty [F_n+G_n] = \sum_0^\infty [F_n+G_{n+1}] = \sum_0^\infty [F_n+\mathscr{M}F_n],$$

and $F_n+\mathscr{M}F_n$ satisfies the boundary condition on the cylinder, the sum f of the convergent series satisfies both the free-surface condition (2.6) and the cylinder condition (2.1) and is therefore the solution to the linear problem.

We shall see that it is not strictly necessary to determine the $\mathscr{M}$ and $\mathscr{K}$ transformations completely at each stage provided it is assumed from the outset that a itself is a small parameter with $h = O(1)$, but nevertheless the Wehausen scheme represents a definite prescription for a solution to the wholly linear problem in any physical situation for which it is convergent, and it is capable of immediate extension to treat the case of non-circular cylinders. But it is also a simple matter to modify the scheme to take account of the pressures $p_n(x)$ of equation (2.10) and hence to treat systematically the full non-linear problem.

Now the potential due to a pressure distribution $p_n(x)$ on the free surface is given by Wehausen & Laitone (1960, p. 601) in the form

$$F_n^{(p)}(z) = -\frac{i}{\pi}\int_{-\infty}^{\infty} ds p_n(s)\, e^{-i(z-ih-s)}\,\mathrm{Ei}^-(i(z-ih-s)), \tag{4.7}$$

and hence if $G_n+\mathscr{K}G_n$ satisfies the homogeneous free-surface condition (2.6) then $G_n+\mathscr{K}G_n+F_n^{(p)}$ satisfies the correct nth-order inhomogeneous free-surface condition (2.10). That is, the Wehausen scheme may be extended to treat the non-linear case merely by adding $F_n^{(p)}$ to the second column, or symbolically by defining a modified Kochin operator $\mathscr{K}'$ such that

$$\mathscr{K}'G_n = \mathscr{K}G_n+F_n^{(p)},$$

(this is only a symbolic relationship since of course $F_n^{(p)}$ can only be calculated from a knowledge of *all* lower approximations, not just G_n itself) and using $\mathscr{K}'$ instead of $\mathscr{K}$ in the scheme (4.6).

The second column of (4.6) has a separate interpretation which is important, for if we define

$$F(z) = \sum_{n=0}^{\infty} F_n(z), \tag{4.8}$$

then clearly

$$f(z) = F(z)+\mathscr{M}F(z).$$

But all $F_n(z)$ are analytic everywhere below the free surface and in particular at the origin, and hence within some circle of convergence which includes the circle, we can expand $F(z)$ in a Taylor series

$$F(z) = \sum_{m=0}^{\infty} C_m z^m, \tag{4.9}$$

for some sequence $C_m = C_m(a, h)$ of complex numbers. Thus in the neighbourhood of the cylinder the complete solution $f(z)$ can be written as

$$f(z) = \sum_{m=0}^{\infty} \left(C_m z^m + \bar{C}_m \frac{a^{2m}}{z^m} \right). \tag{4.10}$$

By an application of Blasius theorem to the potential in this form, Havelock (1936) showed that the forces on the circle are given by

$$X_F - iY_F = 2\pi\rho U^2 \sum_{m=1}^{\infty} m(m+1) a^{2m} \bar{C}_m C_{m+1}. \tag{4.11}$$

Under the conditions in which the Wehausen scheme converges we should expect this series for the forces to converge; in any case for small a the series consists of terms of decreasing order of magnitude with respect to a in the asymptotic sense.

The problem resolves itself therefore to the determination of a sequence of complex co-efficients C_m, each of which can be written as a series, say,

$$C_m = \sum_{m=0}^{\infty} C_{mn},$$

where C_{mn} represents the contribution to the coefficient of z^m from the approximation $F_n(z)$ of order n in the Wehausen scheme. The force is thus obtained as a doubly-infinite series; under the assumption that a is small it may be re-ordered into a single series of decreasing terms as given by Havelock (1936) for the linear case. We may note on inspection of Havelock's result, however, that while the series has the appearance of being ordered with respect to small a and was indeed derived upon that assumption, the coefficients are functions of h of such a character that the series is actually ordered with respect to the combination a^2/h. That is, Havelock has achieved a little more than he set out to, since a series derived upon the assumption that a is small with $h = O(1)$ has turned out to be valid also if h is large. We should not expect that the inclusion of non-linear effects would disturb this qualitative conclusion, since clearly the larger h is (even for a fixed value of a) the more gentle will be the waves and the less important the non-linearity. Hence in the following section we carry through the analysis systematically for the case a small, $h = O(1)$, but expect the theory to be valid also under some circumstances when h is large.

5. Expansion with a as small parameter

Now if we proceed step by step through the modified Wehausen scheme (4.6) we find successively

$$\left.\begin{aligned} &G_0 = 0, \quad F_0 = z, \quad f_0 = z, \\ &G_1 = a^2/z, \quad F_1 = \mathscr{K}(a^2/z) + F_1^{(p)} = a^2\mathscr{K}(1/z) \\ &\qquad = a^2\left(-\frac{1}{z-2ih} + 2i\,e^{-2h-iz}\,\mathrm{Ei}^-(2h+iz)\right), \\ &f_1 = a^2/z + a^2\left(-\frac{1}{z-2ih} + 2i\,e^{-2h-iz}\,\mathrm{Ei}^-(2h+iz)\right), \end{aligned}\right\} \tag{5.1}$$

which is the first approximation (3.1) of Lamb.

Now according to the full Wehausen scheme we should at this stage compute $G_2(z) = \mathscr{M}F_1(z) = \overline{F}_1(a^2/z)$ and take its Kochin transform in full. However, the Wehausen scheme in its full generality is not needed here since we know that a is small, so that the whole analysis may be carried out in the small region $z = O(a)$. Hence it suffices in the Wehausen scheme to use for F_1 the first non-constant term in its Taylor series (constant terms may of course always be ignored without loss of generality). Hence from (4.4)

$$F_1(z) = -a^2q_1 z + O(a^2z^2), \tag{5.2}$$

with the error term being $O(a^4)$ as long as $z = O(a)$.

Now
$$G_2(z) = \mathscr{M}F_1(z) = -\frac{a^4\overline{q}_1}{z} + O\left(\frac{a^6}{z^2}\right), \tag{5.3}$$

and
$$\mathscr{K}G_2 = -a^4q_1\mathscr{K}(1/z) + O(a^6\mathscr{K}(1/z^2)).$$

But by differentation of

$$\mathscr{K}(1/z) = -q_1 z + \tfrac{1}{2}iq_2 z^2 + \ldots,$$

we have
$$\mathscr{K}(1/z^2) = -iq_2 z + \ldots;$$

(it is easy to prove from (4.2) that $\mathscr{K}$ and d/dz commute). Therefore

$$\mathscr{K}G_2 = a^4q_1^2 z - \tfrac{1}{2}ia^4q_1q_2z^2 + O(a^7). \tag{5.4}$$

In order to find F_2 we must add to $\mathscr{K}G_2$ the pressure term

$$F_2^{(p)} = -\frac{i}{\pi}\int_{-\infty}^{\infty} ds\,p_2(s)\,e^{-i(z-ih-s)}\,\mathrm{Ei}^-(i(z-ih-s)). \tag{5.5}$$

Since f_1 is proportional to a^2 and $p_2(x)$ is quadratic in f_1, $F_2^{(p)}$ is proportional to a^4 and we may write

$$F_2^{(p)}(z) = a^4\sum_{m=0}^{\infty}\frac{\gamma_m}{m!}z^m, \tag{5.6}$$

for some Taylor coefficients $\gamma_n = \gamma_n(h)$. In particular the only coefficient we shall need to use is

$$\gamma_2 = -\frac{i}{\pi}\int_{-\infty}^{\infty} dx\,\frac{p_2(x)}{a^4}\left[\frac{1}{(h-ix)^2} + \frac{1}{h-ix} - e^{-h+ix}\,\mathrm{Ei}^-(h-ix)\right]. \tag{5.7}$$

From the definition (2.12) of $p_2(x)$ and the value (3.1) of f_1, we can obtain for the pressure

$$\frac{p_2(x)}{a^4} = -\frac{1}{2}\left|-\frac{1}{(x+ih)^2}+\frac{1}{(x-ih)^2}+\frac{2i}{x-ih}+2\,e^{-h-ix}\,\mathrm{Ei}^-(h+ix)\right|^2$$
$$+\left[\frac{4h(3x^2-h^2)}{(x^2+h^2)^3}+\frac{2(x^2-h^2)}{(x^2+h^2)^2}\right]\mathscr{I}\left(\frac{1}{x+ih}-\frac{1}{x-ih}+2i\,e^{-h-ix}\,\mathrm{Ei}^-(h+ix)\right). \quad (5.8)$$

Using again the series representation (3.3) of the exponential integral this apparently complicated expression may be evaluated easily on a computer, and examples of the pressures for the cases $h = 1$ and $h = 4$ (Froude numbers $h^{-\frac{1}{2}}$ of 1 and $\frac{1}{2}$ respectively, based on depth) are shown in figure 2. Notice that in spite

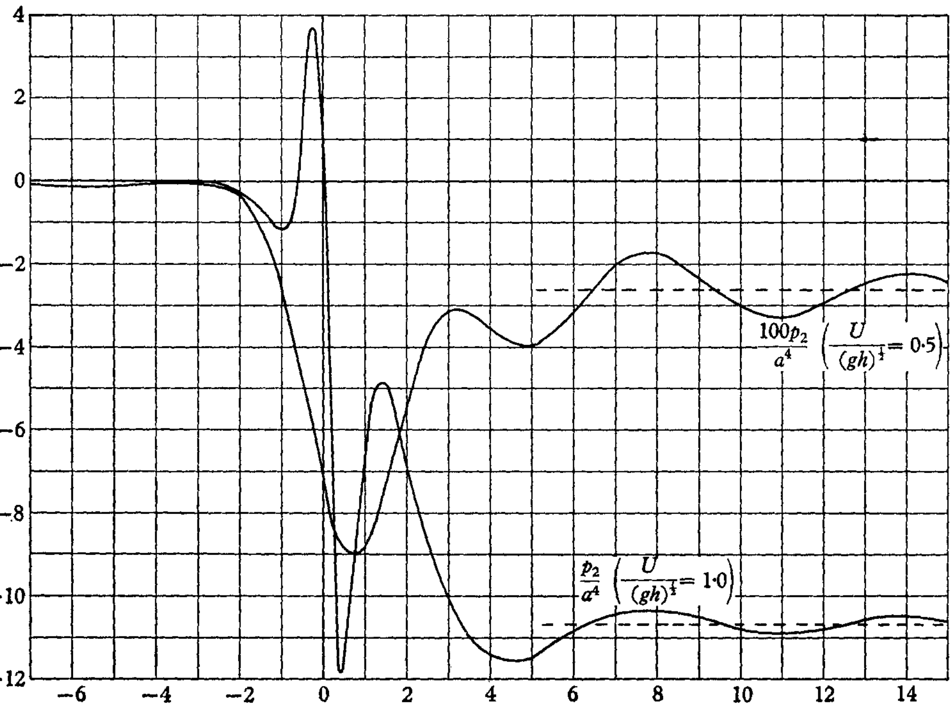

FIGURE 2. The pressure distribution on the free surface obtained from the first-order linearized flow.

of the fact that it is the pressure due to the first-order waves, $p_2(x)$ is not itself sinusoidal far downstream; its oscillations decay (slowly) and p_2 tends to a constant value.† Notice also how much smaller is the pressure at $h = 4$ than at $h = 1$ (the scales differ by a factor of 100) confirming that the non-linear effects tend to zero quite strongly as $h \to \infty$ as well as when $a \to 0$. Another way of expressing this fact is the physically obvious statement that at constant depth the wave height increases very rapidly with increasing speed.

† This constant value is exactly of the right magnitude to balance the 'D.C.' part of the term η_1^2 in the formula (2.9) for the second-order wave height, leaving a pure second-harmonic wave.

The real and imaginary parts of γ_2 may now be found by numerical integration of equation (5.7). In performing this integration one need not be too concerned about curtailing the infinite range of integration; since it is easy to see that the integrand tends to zero rapidly at *both* ends of the range. Physically γ_2 represents a disturbance velocity at the (small) cylinder due to the pressure $p_2(x)$; regions far ahead of the cylinder contribute little because there are no waves and hence small pressure, while regions far behind the cylinder contribute little because the disturbance due to the waves is mainly propagated downstream and not back upstream to the cylinder.

Now we have

$$\begin{aligned} F_2 &= \mathscr{K}G_2 + F_2^{(p)} \\ &= a^4(q_1^2+\gamma_1)z + \tfrac{1}{2}a^4(-iq_1q_2+\gamma_2)z^2 + \dots \end{aligned} \tag{5.9}$$

This is as far as we need go in the Wehausen scheme except to observe that continuing one cycle further would give

$$F_3 = O(a^6)z + O(a^6)z^2 + \dots \tag{5.10}$$

Thus

$$\begin{aligned} F(z) = z & \\ -a^2q_1z \qquad & + \tfrac{1}{2}a^2iq_2z^2 \qquad\qquad + \sum_3^\infty O(a^2)z^m \\ + a^4(q_1^2+\gamma_1)z & + \tfrac{1}{2}a^4(-iq_1q_2+\gamma_2)z^2 + \sum_3^\infty O(a^4)z^m \\ & \qquad\qquad\qquad\qquad + \sum_1^\infty O(a^6)z^m, \end{aligned}$$

giving the following expressions for the coefficients C_m of equation (4.9):

$$\begin{aligned} C_1 &= 1 - a^2q_1 + O(a^4), \\ C_2 &= \tfrac{1}{2}ia^2(q_2 - a^2(q_1q_2 + i\gamma_2)) + O(a^6), \\ C_m &= O(a^2) \quad (m \geqslant 2). \end{aligned}$$

Hence the forces on the cylinder follow from equation (4.11) as

$$\begin{aligned} X_F - iY_F &= 2\pi i\rho a^4U^2[(1-a^2\bar{q}_1 + O(a^4))(q_2 - a^2(q_1q_2+i\gamma_2) + O(a^4)) + O(a^4)] \\ &= 2\pi i\rho a^4U^2[q_2 - a^2(q_1q_2 + \bar{q}_1q_2 + i\gamma_2) + O(a^4)]. \end{aligned} \tag{5.11}$$

Separately the X- and Y-forces are thus:

$$\frac{X_F}{\pi\rho a^2U^2} = -2a^2[\mathscr{I}q_2 - a^2(2\mathscr{R}q_1\mathscr{I}q_2 + \mathscr{R}\gamma_2) + O(a^4)], \tag{5.12}$$

$$\frac{Y_F}{\pi\rho a^2U^2} = -2a^2[\mathscr{R}q_2 - a^2(2\mathscr{R}q_1\mathscr{R}q_2 - \mathscr{I}\gamma_2) + O(a^4)]. \tag{5.13}$$

The left-hand sides of equations (5.12) and (5.13) represent the forces divided by the buoyancy, $\rho g\pi a^2$, of the cylinder per unit length (since in our units $U^2 = g$) so that the right-hand sides are the quantities plotted by Havelock. Except for the terms in γ_2, (5.12) and (5.13) are identical with Havelock's (1936, p. 532) equations (29) and (30) carried as far as the 2nd approximation; Havelock's equations in fact go as far as the 5th approximation for the linear problem.

Bessho (1957) carried through an analysis of the second-order non-linear potential with results (for the potential) not inconsistent with ours. However, he then obtained the forces by use of a formula of Havelock involving the wave amplitude far downstream, instead of by Blasius theorem as used here. Since Havelock's formula is only valid in the linearized approximation, Bessho was led to an erroneous conclusion that, although non-linearity does affect the potential at second order, it only enters the forces at third order. The correct result could be obtained from Bessho's potential by sufficiently careful consideration of energy radiation to infinity, taking account of second-harmonic contributions, but it appears easier to proceed via Blasius theorem.

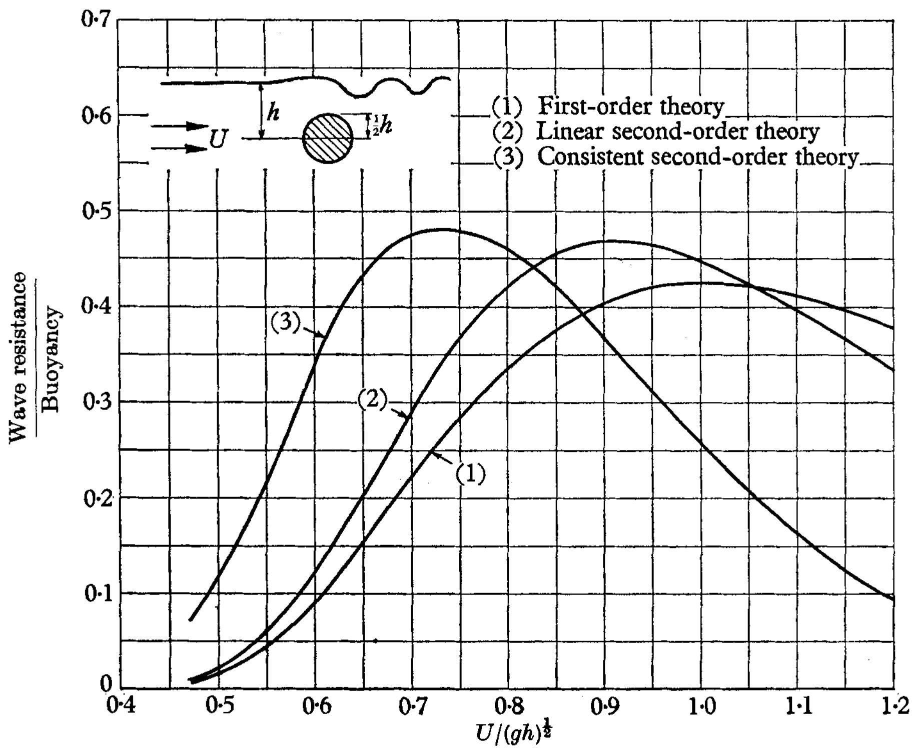

FIGURE 3.

Since we are now able to compute the real and imaginary parts of the non-linear contribution γ_2, it is possible to compare the relative importance of the two second-order effects. Notice that since the q's and γ's are not functions of a, this comparison is independent of the value or even the order of magnitude of a and is only affected by the value of h. Clearly a determines how important second-order effects are as a whole and how fast (if at all) the series begun by (5.12) and (5.13) converge, but does not affect the *relative* importance of the two second-order effects. In figures 3 and 4 we plot (1) the first approximations obtained by using the first terms only, (2) the Havelock second approximation consisting of the first two terms in equations (5.12), (5.13), and finally (3) the complete second-order forces, for the case $a/h = \frac{1}{2}$.

It must be emphasized that we do not pretend that, for such a severe disturbance, any of the curves in figures 3 and 4 give good approximations to the wave forces on the cylinder—much less that they are directly relevant to the real physical problem of flow past a cylinder, for which even the exact potential theory is a poor model. Our purpose is simply to show how the non-linear second-order effect dominates the linear second-order effect over the complete

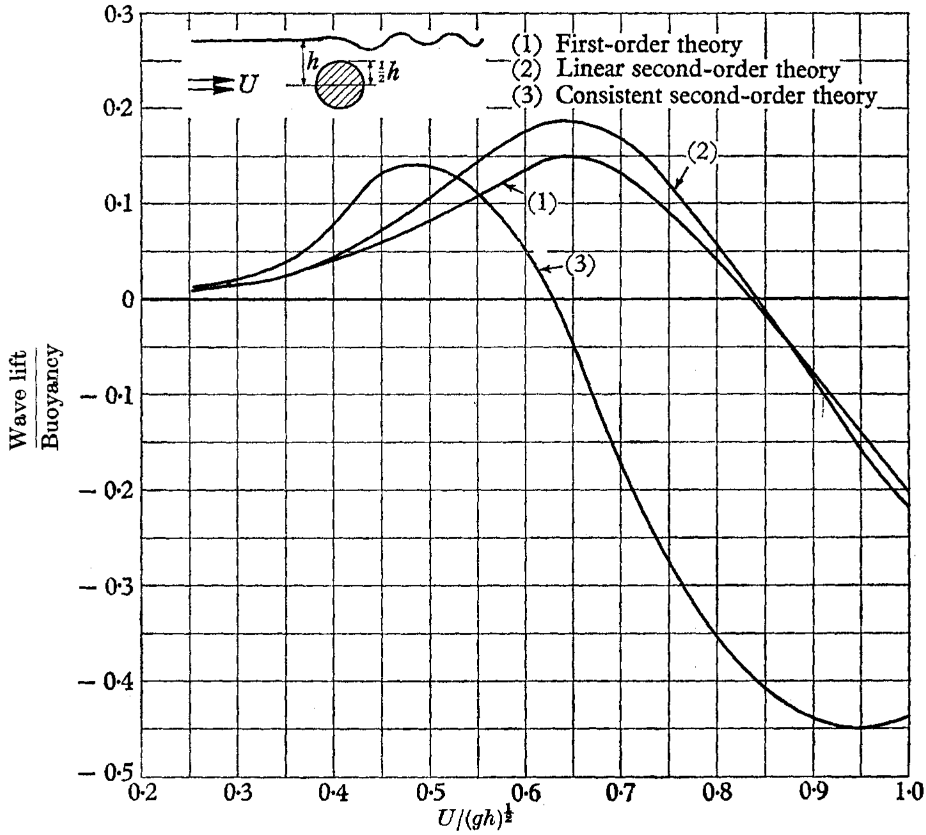

FIGURE 4.

range of speeds, and this can be shown most clearly by presenting a case in which both second-order effects are large. However, for comparison we also give in figure 5 the horizontal force for the less severe case $a/h = \frac{1}{4}$ where second-order effects are smaller and we should hope that the full second-order curve is a good approximation to the true wave resistance. From (5.12) it is clear that, at any fixed abscissa, the ratio [(3)–(2)]/[(2)–(1)] is the same in figures 3 and 5; that is, as explained above, the *relative* importance of the two second-order effects is independent of the severity of the disturbance.

The curves show clearly that, by a factor of at least 2 or 3 in the range of Froude numbers where wave-making is significant, it is more important to correct for non-linearity at the free surface than for the fact that the boundary condition is not satisfied exactly by the first approximation on the body surface. It is tempting to generalize this argument to a wider range of water-wave problems, with the conclusion that much of present-day effort towards theoretical prediction of ship behaviour is mis-directed, but this temptation should be resisted. The argument only applies with certainty to the class of steady two-dimensional deep-submergence problems considered here, or perhaps (with a little less certainty—it would be interesting to repeat the present analysis for (say) a submerged sphere to check this) to the whole class of problems labelled (C) in the introduction. The extension to problems of class (B) is less certain, but this paper may serve as a warning that non-linear effects can be at least as important as other inaccuracies in the standard theoretical procedures for solving water-wave problems involving flows past rigid bodies.

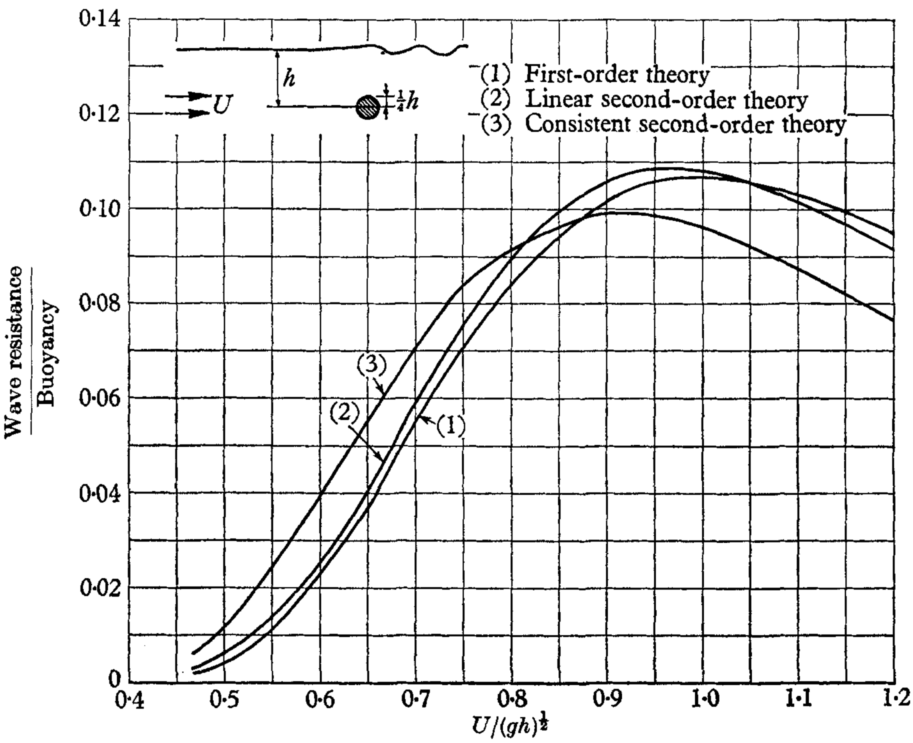

FIGURE 5.

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